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On some explicit evaluations of multiple zeta-star values

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Abstract

In this paper, we give some explicit evaluations of multiple zeta-star values which are rational multiple of powers of π^2.

Keywords: Multiple zeta value; Multiple zeta-star value

1. Main results

The multiple zeta value (MZV) is defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) := \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

where $k_1, k_2, \ldots, k_n$ are positive integers and $k_1 \geqslant 2$. The integers $k = k_1 + k_2 + \cdots + k_n$ and n are called weight and depth respectively. Considerable amount of work on MZV's has been done in recent years from various aspects and interests. Among them, several explicit values are known for special index sets, as will be recalled below.

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In this paper, we give some evaluations of the *multiple zeta-star value* (MZSV), which is defined by the following series similar to the MZV:

$$\zeta^*(k_1, k_2, \dots, k_n) := \sum_{m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

where $k_1, k_2, \dots, k_n$ satisfy the same condition as above. The MZSV can be expressed as a $\mathbb{Z}$-linear combination of MZV's, and vice versa.

Theorem A. *For positive integers m, n, we have*

$$\zeta^*(\underbrace{2m, 2m, \dots, 2m}_{n}) = \left\{ \sum_{\substack{n_0+\cdots+n_{m-1}=mn \\ n_i \geqslant 0}} (-1)^{m(n-1)} \left(\prod_{k=0}^{m-1} \frac{(2^{2n_k}-2)B_{2n_k}}{(2n_k)!} \right) \exp\left(\frac{2\pi i}{m} \sum_{l=0}^{m-1} l n_l \right) \right\} \pi^{2mn}.$$

Theorem B. *For positive integer n, we have*

$$\zeta^*(\underbrace{3, 1, \dots, 3, 1}_{2n}) = \sum_{i=0}^{n} \left\{ \frac{2}{(4i+2)!} \sum_{\substack{n_0+n_1=2(n-i) \\ n_0, n_1 \geqslant 0}} (-1)^{n_1} \frac{(2^{2n_0}-2)B_{2n_0}}{(2n_0)!} \frac{(2^{2n_1}-2)B_{2n_1}}{(2n_1)!} \right\} \pi^{4n}.$$

In particular,

$$\zeta^*(\underbrace{3, 1, \dots, 3, 1}_{2n}) \in \mathbb{Q} \times \pi^{4n}.$$

Theorem C. *Let n be a positive integer, and let I_{2n} denote the set of all $2n+1$ possible insertions of the number 2 in the string $\{\underbrace{3, 1, \dots, 3, 1}_{2n}\}$. Then we have*

$$\sum_{\vec{s}_{2n} \in I_{2n}} \zeta^*(\vec{s}_{2n}) = \sum_{k=0}^{n} \left\{ \frac{2^{4k+3} B_{4k+2}}{(4k+2)!} \sum_{i=0}^{n-k} \frac{\alpha_{n-k-i}}{(4i+2)!} - \frac{\alpha_{n-k}}{(4k+3)!} \right\} \pi^{4n+2},$$

where

$$\alpha_n = \sum_{\substack{n_0+n_1=2n \\ n_0, n_1 \geqslant 0}} (-1)^{n_1} \frac{(2^{2n_0}-2)B_{2n_0}}{(2n_0)!} \frac{(2^{2n_1}-2)B_{2n_1}}{(2n_1)!}.$$

In particular,

$$\sum_{\vec{s}_{2n}\in I_{2n}} \zeta^*(\vec{s}_{2n}) \in \mathbb{Q}\times\pi^{4n+2}.$$

For later use, we recall the corresponding results for MZV's.

Theorem 1. *(See [1].) Let m,n be positive integers. Then we have*

$$\zeta(\underbrace{2m,2m,\ldots,2m}_{n}) = C_n^{(m)}\frac{(2\pi i)^{2mn}}{(2mn)!},$$

where $C_n^{(m)}$ is defined by the following recurrence relations:

$$C_0^{(m)} = 1, \qquad C_n^{(m)} = \frac{1}{2n}\sum_{l=1}^{n}(-1)^l\binom{2mn}{2ml}B_{2ml}C_{n-l}^{(m)} \quad (n\geqslant 1),$$

with B_{2n} being the classical Bernoulli numbers.

Theorem 2. *(See [2,3].) For any positive integer n, we have*

$$\zeta(\underbrace{3,1,\ldots,3,1}_{2n}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

Theorem 3. *(See [2].) Let n be a positive integer, and let I_{2n} denote the set of all $2n+1$ possible insertions of the number 2 in the string $\{\underbrace{3,1,\ldots,3,1}_{2n}\}$. Then*

$$\sum_{\vec{s}_{2n}\in I_{2n}} \zeta(\vec{s}_{2n}) = \frac{\pi^{4n+2}}{(4n+3)!}.$$

2. Algebraic setup

We use the algebraic setup of MZV's that was developed by Hoffman [4]. Consider the non-commutative polynomial ring

$$\mathfrak{H} := \mathbb{Q}\langle x,y\rangle$$

in two indeterminates x, y. We refer to monomials in x and y as words. We also define subrings

$$\mathfrak{H}^1 := \mathbb{Q} + \mathfrak{H}y$$

and

$$\mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y.$$

For an integer $k \geqslant 1$, put $z_k = x^{k-1}y$. Then the ring $\mathfrak{H}^1$ is freely generated by z_k $(k = 1, 2, 3, \ldots)$. When $k \geqslant 2$, z_k is contained $\mathfrak{H}^0$.

Now define the evaluation map $Z : \mathfrak{H}^0 \to \mathbb{R}$ by setting

$$Z(z_{k_1} z_{k_2} \cdots z_{k_n}) = \zeta(k_1, k_2, \ldots, k_n)$$

on generators and extending it $\mathbb{Q}$-linearly.

We define the harmonic product $*$ on $\mathfrak{H}^1$ inductively by

$$w * 1 = 1 * w = w,$$

$$z_p w_1 * z_q w_2 = z_p(w_1 * z_q w_2) + z_q(z_p w_1 * w_2) + z_{p+q}(w_1 * w_2),$$

for all $p, q \geqslant 1$, and any words $w, w_1, w_2 \in \mathfrak{H}^1$, together with $\mathbb{Q}$-bilinearity. For instance, $z_p * z_q = z_p z_q + z_q z_p + z_{p+q}$. This product corresponds to $\zeta(p)\zeta(q) = \zeta(p, q) + \zeta(q, p) + \zeta(p+q)$.

The following theorem which has been proven in [4] gives the basic algebraic properties of the $*$-product.

Theorem 4. *(See [4].) The harmonic product is commutative and associative.*

Theorem 4 says that $\mathfrak{H}^1$ is a $\mathbb{Q}$-commutative algebra with respect to the harmonic product $*$. Then $\mathfrak{H}^0$ is subalgebra of $\mathfrak{H}^1$. In [4], it has also been proven that Z is homomorphism with respect to the harmonic product $*$:

$$Z(w_1 * w_2) = Z(w_1)Z(w_2) \quad (w_1, w_2 \in \mathfrak{H}^0).$$

We conclude this section by introducing the $\mathbb{Q}$-linear map S. Let $S_1 \in \mathrm{Aut}(\mathfrak{H})$ be defined by $S_1(1) = 1$, $S_1(x) = x$ and $S_1(y) = x + y$. Define the $\mathbb{Q}$-linear map $S : \mathfrak{H}^1 \to \mathfrak{H}^1$ by

$$S(Fy) := S_1(F)y$$

for all words $F \in \mathfrak{H}$ and $S(1) = 1$. Then it is clear that

$$\zeta^*(k_1, k_2, \ldots, k_n) = Z\bigl(S(z_{k_1} z_{k_2} \cdots z_{k_n})\bigr).$$

For example, $\zeta^*(k_1, k_2) = \zeta(k_1 + k_2) + \zeta(k_1, k_2) = Z(S(z_{k_1} z_{k_2}))$, $\zeta^*(k_1, k_2, k_3) = \zeta(k_1 + k_2 + k_3) + \zeta(k_1 + k_2, k_3) + \zeta(k_1, k_2 + k_3) + \zeta(k_1, k_2, k_3) = Z(S(z_{k_1} z_{k_2} z_{k_3}))$.

3. Proof of Theorem A

We prove Theorem A by using the Laurent expansion for the cosecant function:

$$\csc x = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2^{2n} - 2)B_{2n}}{(2n)!} x^{2n-1}.$$

Proof of Theorem A. Using the infinite product for the sine function, we have

$$\csc \pi e^{\frac{\pi i}{m}k} x = \frac{1}{\pi e^{\frac{\pi i}{m}k} x \prod_{n=1}^{\infty} \bigl(1 - e^{\frac{2\pi i}{m}k} \frac{x^2}{n^2}\bigr)}.$$

Substituting $k = 0, 1, \dots, m-1$ and multiplying both sides, we obtain

$$\left(\pi^m x^m \prod_{k=0}^{m-1} e^{\frac{\pi i}{m}k}\right) \prod_{k=0}^{m-1} \csc \pi e^{\frac{\pi i}{m}k} x = \frac{1}{\prod_{n=1}^{\infty}\left(1 - \frac{x^{2m}}{n^{2m}}\right)}. \tag{1}$$

The right-hand side of (1) equals

$$1 + \left(\sum_{n_1>0} \frac{1}{n_1^{2m}}\right)x^{2m} + \left(\sum_{n_1 \geqslant n_2 > 0} \frac{1}{n_1^{2m} n_2^{2m}}\right)x^{4m} + \cdots = 1 + \sum_{n=1}^{\infty} \zeta^*(\underbrace{2m, 2m, \dots, 2m}_{n})x^{2mn}.$$

On the other hand, the left-hand side of (1) equals

$$\begin{aligned}
&\left(\pi^m x^m \prod_{k=0}^{m-1} e^{\frac{\pi i}{m}k}\right) \prod_{k=0}^{m-1} \sum_{n_k=0}^{\infty} (-1)^{n_k - 1} \frac{(2^{2n_k} - 2)B_{2n_k}}{(2n_k)!} \pi^{2n_k - 1} e^{\frac{\pi i}{m}k(2n_k - 1)} x^{2n_k - 1} \\
&= \prod_{k=0}^{m-1} \sum_{n_k=0}^{\infty} (-1)^{n_k - 1} \frac{(2^{2n_k} - 2)B_{2n_k}}{(2n_k)!} \pi^{2n_k} e^{\frac{2\pi i}{m}kn_k} x^{2n_k} \\
&= 1 + \sum_{n=1}^{\infty} \left\{ \sum_{\substack{n_0 + \cdots + n_{m-1} = mn \\ n_i \geqslant 0}} (-1)^{m(n-1)} \left(\prod_{k=0}^{m-1} \frac{(2^{2n_k} - 2)B_{2n_k}}{(2n_k)!}\right) \right. \\
&\quad \left. \times \exp\left(\frac{2\pi i}{m} \sum_{l=0}^{m-1} l n_l\right)\right\} \pi^{2mn} x^{2mn}.
\end{aligned}$$

Comparing coefficients of both sides, we obtain the desired identity. □

Corollary 5. *For positive integers m,n, we have*

$$\zeta^*(\underbrace{2m, 2m, \dots, 2m}_{n}) \in \mathbb{Q} \times \pi^{2mn}.$$

Proof. The coefficient of π^{2mn} on the right-hand side of Theorem A is invariant under the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, hence belongs to $\mathbb{Q}$. □

Remark 1. Yasuo Ohno proved Theorem A independently. He proved this theorem in two ways, one way is to use the same method of our proof. The other is to use generating function and differential equation.

Remark 2. In [5], Sergey Zlobin obtained the same generating function as (1) of multiple zeta-star values and gave the relation

$$\zeta^*(\underbrace{2, 2, \dots, 2}_{n}) = 2(1 - 2^{1-2n})\zeta(2n).$$

4. Proof of Theorem B

Theorem B will be obtained as a corollary of a more general identity, which is stated as follows.

Theorem 6. *For positive integers a, b and nonnegative integer n, we have*

$$S\big((z_a z_b)^n\big) = \sum_{i=0}^{n} (z_a z_b)^i * S\big(z_{a+b}^{n-i}\big), \tag{2}$$

$$S\big(z_b (z_a z_b)^n\big) = \sum_{i=0}^{n} z_b (z_a z_b)^i * S\big(z_{a+b}^{n-i}\big). \tag{3}$$

Proof. By the definition of S, we have

$$S(w_1 w_2) = S_1(w_1) S(w_2) \quad (w_1 \in \mathfrak{H}, w_2 \in \mathfrak{H}^1).$$

Using this identity, we obtain

$$\begin{aligned} S(z_{k_1} z_{k_2} \cdots z_{k_n}) &= z_{k_1} S(z_{k_2} z_{k_3} \cdots z_{k_n}) + S(z_{k_1+k_2} z_{k_3} \cdots z_{k_n}) \\ &= z_{k_1} S(z_{k_2} z_{k_3} \cdots z_{k_n}) + z_{k_1+k_2} S(z_{k_3} \cdots z_{k_n}) + S(z_{k_1+k_2+k_3} z_{k_4} \cdots z_{k_n}) \\ &= \cdots \\ &= \sum_{j=1}^{n} z_{k_1+k_2+\cdots+k_j} S(z_{k_{j+1}} z_{k_{j+2}} \cdots z_{k_n}). \end{aligned} \tag{4}$$

(When $j = n$, we regard $S(z_{k_{j+1}} z_{k_{j+2}} \cdots z_{k_n})$ as 1.) We prove identities (2) and (3) simultaneously by induction. The case of $n = 0$ is obvious. Suppose that the assertion has been proven up to $n-1$.

$$\begin{aligned} \text{RHS of (2)} &\overset{(4)}{=} S\big(z_{a+b}^n\big) + \sum_{i=1}^{n-1} (z_a z_b)^i * \sum_{j=1}^{n-i} z_{(a+b)j} S\big(z_{a+b}^{n-i-j}\big) + (z_a z_b)^n \\ &= S\big(z_{a+b}^n\big) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} z_a \big(z_b (z_a z_b)^{i-1} * z_{(a+b)j} S\big(z_{a+b}^{n-i-j}\big)\big) \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} z_{(a+b)j} \big((z_a z_b)^i * S\big(z_{a+b}^{n-i-j}\big)\big) \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} z_{(a+b)j+a} \big(z_b (z_a z_b)^{i-1} * S\big(z_{a+b}^{n-i-j}\big)\big) + (z_a z_b)^n \\ &\overset{(4)}{=} S\big(z_{a+b}^n\big) + z_a \sum_{i=1}^{n-1} z_b (z_a z_b)^{i-1} * S\big(z_{a+b}^{n-i}\big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n-1} z_{(a+b)j} \sum_{i=1}^{n-j} (z_a z_b)^i * S\bigl(z_{a+b}^{n-j-i}\bigr) \\
&+\sum_{j=1}^{n-1} z_{(a+b)j+a} \sum_{i=1}^{n-j} z_b (z_a z_b)^{i-1} * S\bigl(z_{a+b}^{n-j-i}\bigr) + (z_a z_b)^n \\
= {}& S\bigl(z_{a+b}^n\bigr) + z_a\bigl\{S\bigl(z_b(z_a z_b)^{n-1}\bigr) - z_b(z_a z_b)^{n-1}\bigr\} \\
&+\sum_{j=1}^{n-1} z_{(a+b)j}\bigl\{S\bigl((z_a z_b)^{n-j}\bigr) - S\bigl(z_{a+b}^{n-j}\bigr)\bigr\} \\
&+\sum_{j=1}^{n-1} z_{(a+b)j+a} S\bigl(z_b(z_a z_b)^{n-j-1}\bigr) + (z_a z_b)^n \\
&\text{(by induction hypothesis)} \\
\overset{(4)}{=} {}& S\bigl(z_{a+b}^n\bigr) + z_a S\bigl(z_b(z_a z_b)^{n-1}\bigr) + \sum_{j=1}^{n-1} z_{(a+b)j} S\bigl((z_a z_b)^{n-j}\bigr) \\
&- \bigl\{S\bigl(z_{a+b}^n\bigr) - z_{(a+b)n}\bigr\} + \sum_{j=1}^{n-1} z_{(a+b)j+a} S\bigl(z_b(z_a z_b)^{n-j-1}\bigr) \\
= {}& \sum_{j=0}^{n-1} z_{(a+b)j+a} S\bigl(z_b(z_a z_b)^{n-j-1}\bigr) + \sum_{j=1}^{n} z_{(a+b)j} S\bigl((z_a z_b)^{n-j}\bigr).
\end{aligned}
$$

By (4), this equals to LHS of (2). In the same way, (3) can be proved by using the induction hypothesis and (2) for n. □

Proof of Theorem B. From (2), we have

$$\zeta^*(\underbrace{3,1,\ldots,3,1}_{2n}) = \sum_{i=0}^{n} \zeta(\underbrace{3,1,\ldots,3,1}_{2i})\zeta^*(\underbrace{4,4,\ldots,4}_{n-i}).$$

Hence, we have the assertion by Theorem 2 and Theorem A. □

5. Proof of Theorem C

As in Section 4, we prove the following identities to obtain the explicit evaluations of $\sum_{\vec{s}_{2n}\in I_{2n}} \zeta^*(\vec{s}_{2n})$.

Theorem 7. *We put $A_{i,j} = (z_a z_b)^i z_c (z_a z_b)^j$ and $B_{i,j} = (z_b z_a)^i z_c (z_b z_a)^j z_b$. For positive integers a, b, c and nonnegative integer n, we have*

$$\sum_{k=0}^{n} S(A_{k,n-k}) + \sum_{k=0}^{n-1} S(z_a B_{k,n-1-k}) = 2\sum_{k=0}^{n} z_{(a+b)k+c} * S\big((z_a z_b)^{n-k}\big) - \sum_{k=0}^{n} S\big(z_{a+b}^{n-k}\big) * \left\{ \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right\} \tag{5}$$

and

$$\sum_{k=0}^{n} S(z_b A_{k,n-k}) + \sum_{k=0}^{n} S(B_{k,n-k}) = 2\sum_{k=0}^{n} z_{(a+b)k+c} * S\big(z_b (z_a z_b)^{n-k}\big) - \sum_{k=0}^{n} S\big(z_{a+b}^{n-k}\big) * \left\{ \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right\}. \tag{6}$$

(*We regard summations* $\sum_{i=m}^{m-1} \ldots$ *as* 0.)

Proof. We prove the identities (5) and (6) simultaneously by induction. The case of $n = 0$ is obvious. Suppose that the assertion has been proven up to $n - 1$. By using (4) and the definition of the harmonic product, we can transform the first sum on the right of (5) as

$$\begin{aligned}
\sum_{k=0}^{n} z_{(a+b)k+c} * S\big((z_a z_b)^{n-k}\big) &= \sum_{k=0}^{n} z_{(a+b)k+c} S\big((z_a z_b)^{n-k}\big) \\
&\quad + \sum_{j=1}^{n} z_{(a+b)(j-1)+a} \sum_{k=0}^{n-j} z_{(a+b)k+c} * S\big(z_b (z_a z_b)^{n-k-j}\big) \\
&\quad + \sum_{j=1}^{n} z_{(a+b)j} \sum_{k=0}^{n-j} z_{(a+b)k+c} * S\big((z_a z_b)^{n-k-j}\big) \\
&\quad + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \big\{ z_{(a+b)(k+j-1)+a+c} S\big(z_b (z_a z_b)^{n-k-j}\big) \\
&\quad + z_{(a+b)(k+j)+c} S\big((z_a z_b)^{n-k-j}\big) \big\}.
\end{aligned}$$

On the other hand, we find in a similar fashion

$$\begin{aligned}
&\sum_{k=0}^{n} S\big(z_{a+b}^{n-k}\big) * \left\{ \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right\} \\
&\quad = \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left\{ S\big(z_{a+b}^{n-k-j}\big) * \left(\sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right) \right\} \\
&\qquad + \sum_{j=1}^{n} z_{(a+b)j} \big(S\big(z_{a+b}^{n-j}\big) * z_c \big)
\end{aligned}$$

$$
\begin{aligned}
&+ z_c \sum_{k=0}^{n-1} S\big(z_{a+b}^{n-k}\big) * (z_a z_b)^k + \sum_{j=1}^{n} z_{(a+b)j+c} \sum_{k=0}^{n-j} S\big(z_{a+b}^{n-k-j}\big) * (z_a z_b)^k \\
&+ \sum_{k=1}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)j+a} \left\{ S\big(z_{a+b}^{n-k-j}\big) * \left(\sum_{i=1}^{k} z_b A_{i-1,k-i} + \sum_{i=0}^{k-1} B_{i,k-1-i} \right) \right\} \\
&+ \sum_{i=0}^{n} A_{i,n-i} + \sum_{i=0}^{n-1} z_a B_{i,n-1-i}.
\end{aligned}
$$

By Theorem 6(2), this is equal to

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left\{ S\big(z_{a+b}^{n-k-j}\big) * \left(\sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right) \right\} \\
&\quad + z_c S\big((z_a z_b)^n\big) - z_c (z_a z_b)^n + \sum_{j=1}^{n} z_{(a+b)j+c} S\big((z_a z_b)^{n-j}\big) \\
&\quad + \sum_{k=1}^{n-1} \sum_{j=1}^{n+1-k} z_{(a+b)(j-1)+a} \left\{ S\big(z_{a+b}^{n-k-j+1}\big) * \left(\sum_{i=0}^{k-1} z_b A_{i,k-1-i} + \sum_{i=0}^{k-1} B_{i,k-1-i} \right) \right\} \\
&\quad + \sum_{i=0}^{n} A_{i,n-i} + \sum_{i=0}^{n-1} z_a B_{i,n-1-i} \\
&= \sum_{j=0}^{n} z_{(a+b)j+c} S\big((z_a z_b)^{n-j}\big) \\
&\quad + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left\{ S\big(z_{a+b}^{n-k-j}\big) * \left(\sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right) \right\} \\
&\quad + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(j-1)+a} \left\{ S\big(z_{a+b}^{n-k-j}\big) * \left(\sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right) \right\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\text{RHS of (5)} = &\sum_{k=0}^{n} z_{(a+b)k+c} S\big((z_a z_b)^{n-k}\big) \\
&+ 2 \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \big\{ z_{(a+b)(k+j-1)+a+c} S\big(z_b (z_a z_b)^{n-k-j}\big) \\
&+ z_{(a+b)(k+j)+c} S\big((z_a z_b)^{n-k-j}\big) \big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} z_{(a+b)j}\left\{2\sum_{k=0}^{n-j} z_{(a+b)k+c} * S\big((z_a z_b)^{n-j-k}\big)\right.\\
&\left.-\sum_{k=0}^{n-j} S\big(z_{a+b}^{n-j-k}\big) * \left(\sum_{i=0}^{k} A_{i,k-i}+\sum_{i=0}^{k-1} z_a B_{i,k-1-i}\right)\right\}\\
&+\sum_{j=1}^{n} z_{(a+b)(j-1)+a}\left\{2\sum_{k=0}^{n-j} z_{(a+b)k+c} * S\big(z_b (z_a z_b)^{n-j-k}\big)\right.\\
&\left.-\sum_{k=0}^{n-j} S\big(z_{a+b}^{n-j-k}\big) * \left(\sum_{i=0}^{k} z_b A_{i,k-i}+\sum_{i=0}^{k} B_{i,k-i}\right)\right\}\\
=&\sum_{k=0}^{n} z_{(a+b)k+c} S\big((z_a z_b)^{n-k}\big)\\
&+2\sum_{k=0}^{n-1}\sum_{j=k+1}^{n}\left\{z_{(a+b)(j-1)+a+c} S\big(z_b (z_a z_b)^{n-j}\big)+z_{(a+b)j+c} S\big((z_a z_b)^{n-j}\big)\right\}\\
&+\sum_{j=1}^{n} z_{(a+b)j}\left\{\sum_{k=0}^{n-j} S(A_{k,n-j-k})+\sum_{k=0}^{n-j-1} S(z_a B_{k,n-j-1-k})\right\}\\
&+\sum_{j=1}^{n} z_{(a+b)(j-1)+a}\left\{\sum_{k=0}^{n-j} S(z_b A_{k,n-j-k})+\sum_{k=0}^{n-j} S(B_{k,n-j-k})\right\}\\
&\text{(by induction hypothesis)}\\
\overset{(4)}{=}&\sum_{k=0}^{n} S(A_{k,n-k})+\sum_{k=0}^{n-1} S(z_a B_{k,n-1-k}).
\end{aligned}
$$

Hence (5) is true for n. In the same way, (6) can be proved by using the induction hypothesis and (5) for n. □

Proof of Theorem C. By (5), we obtain

$$
\sum_{\vec{s}_{2n}\in I_{2n}} \zeta^*(\vec{s}_{2n}) = 2\sum_{k=0}^{n} \zeta(4k+2)\zeta^*(\underbrace{3,1,\ldots,3,1}_{2n-2k}) - \sum_{k=0}^{n} \zeta^*(\underbrace{4,\ldots,4}_{n-k}) \sum_{\vec{s}_{2k}\in I_{2k}} \zeta(\vec{s}_{2k}).
$$

Hence, we have the assertion by Theorems 1, 3, A and B. □

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